Controlling a ray bundle with a free-form reflector

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The problem of controlling a single ray bundle with a single reflector is not generally solvable, but approximate solutions may often be found that are acceptable for applications. We introduce a new technique for finding such approximations and apply it to the design of a driver-side mirror for an automobile that has no blind spot and minimal distortion. © 2008 Optical Society of America

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We consider the geometric optics problem of controlling a ray bundle with a single reflector. By this, we mean that a target surface is given and each ray in the bundle is assigned a point on that target surface to which we wish to guide that ray by a single reflection off of a mirror surface.

A bundle of rays passing through a point may be used to model pinhole imaging. (Owing to the reversibility of light in geometric optics such a bundle may also be used to model light emanating from a point source, i.e., illumination, as, for example, in [1,2].) In most imaging applications though, many bundles must be simultaneously controlled. Yet even in the case of a single bundle the problem is not completely understood. If one tries to control the bundle with a single reflector, then it can be shown that the problem generally does not have an exact solution [3]. The problem of interest addressed here is to find reflector shapes that approximately control a given ray bundle in a prescribed way, since for applications an approximate solution may be adequate. A method for finding such approximations is presented below. Here, the pinhole camera is a satisfactory model of the human eye and we give an application to the design of a curved side-view mirror for an automobile that has no blind spot and no distortion; i.e., the driver is presented with a wide-angle perspective view.

For our application we assume that we are given a ray bundle passing though a point and a target surface $S$ and a mapping, $T$, that assigns to each ray in the bundle a point on $S$, as in Fig. 1. An image plane is chosen, and the rays of the bundle are parameterized by their intersection with the plane. In Fig. 1 we have chosen the source (e.g., the driver’s eye) to be $[0, 0, 0]$ and the image plane to be $z=1$, so a ray passing through the point $[x,y,z]$ uniquely determines the point in the image plane $[x/z, y/z, 1]$. To solve the problem one must find a single reflective surface, $M$, that will take each ray into its target point. (This method does work for refractive surfaces, although one must take care to avoid total internal reflection.)

Given this information, i.e., the ray bundle, image plane, target surface, and $T$, a vector field $N$ can then be defined at every point of the bundle, which will be normal to the required mirror surface $M$, if $M$ exists. Calculating $N$ is straightforward and is illustrated in Fig. 1. One fixes a point $[x,y,z]$, which lies on a ray emanating from the source, and then computes the unit direction back along the ray,

$$\text{back} = \frac{[-x,-y,-z]}{||[-x,-y,-z]||},$$

and adds it to the unit direction from the point on the ray to the target point,

$$\text{forward} = \frac{T(x/z,y/z) - [x,y,z]}{||T(x/z,y/z) - [x,y,z]||}.$$  

Note that if $\nabla \times N = 0$ then $N$ is a gradient; i.e., there is a scalar function $\phi$ such that $N = \nabla \phi$. If this is the case then the level surfaces $\phi(x,y,z) = C$ are perpendicular to $N$ and represent solution surfaces. It could be the case though that $N$ is a multiple of a gradient, in which case surfaces perpendicular to $N$ will exist. In this case it is a necessary condition that $N \cdot (\nabla \times N) = 0$. This is a classical fact of geometric optics, which was used originally to test if a wavefront would exist for a given ray bundle [4]. In most cases though, as in our application, $N \cdot (\nabla \times N) \neq 0$.

The method introduced here for constructing a surface that is approximately normal to $N$ can best be described by considering the case when a solution does exist. Consider a surface $z = f(x,y)$. We will show how to construct sample points on this surface by using the normals to the surface and a differential equation. Fix a direction $[\cos \theta, \sin \theta]$ in the $x-y$ plane. Then the restriction of $f(x,y)$ to the line

![Fig. 1. (Color online) Given a correspondence, $T$, that assigns lines in a bundle to points on a target surface, one can define a vector field $N$ that is hopefully normal to a mirror surface that realizes the correspondence.](316x93 to 557x218)_

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through the origin in this direction gives a curve on the surface. Another way to view this is that we have fixed the polar angle \( \theta \), and the corresponding line is given parametrically by \( r \rightarrow [r \cos \theta, r \sin \theta] \). This line determines a vertical plane, which intersects the surface in a curve, as in Fig. 2. We wish to find a means of computing sample points on the graph of \( f \) using only \([-f_x, -f_y, 1]\), i.e., the normals to the surface. The derivative of the curve on the surface in the radial direction is \( (d/dr)(r \cos \theta, r \sin \theta, 1) \), i.e.,

\[
\frac{dz}{dr} = f_z(r \cos \theta, r \sin \theta) \cos \theta + f_y(r \cos \theta, r \sin \theta) \sin \theta.
\]

(3)

Integrating Eq. (3) with the initial condition \( z(0) = f(0,0) \) will give the curve that results from intersecting the surface with the vertical plane, as depicted in Fig. 2. By choosing many such directions \( \theta \) we can generate a large number of sample points on the surface (integrating numerically if necessary).

For the problem of finding a surface approximately normal to a given \( N \) we scale the vector field \( N(x,y,z) = [N_1(x,y,z), N_2(x,y,z), N_3(x,y,z)] \) by \( 1/N_3 \) to create a new vector field:

\[
W(x,y,z) = [W_1, W_2, 1] = \left[ \frac{N_1}{N_3}, \frac{N_2}{N_3}, 1 \right].
\]

In analogy with Eq. (3), we have a new differential equation:

\[
\frac{dz}{dr} = -W_1(r \cos \theta, r \sin \theta, z(r)) \cos \theta
- W_2(r \cos \theta, r \sin \theta, z(r)) \sin \theta.
\]

(5)

Choosing an initial point \([0,0,h]\) and integrating Eq. (5) for many different choices of \( \theta \) will generate points on a surface. If there exists a surface perpendicular to \( N \) that contains the initial point then the resulting curves will all lie on that surface. Otherwise the curves form a surface that is approximately perpendicular to \( N \). The approximation will be best at the initial condition and deteriorate as the distance to the initial condition increases, but we do not present estimates of this error.

Other techniques for addressing the problem of finding a surface perpendicular to a given vector field appear in [3,5,6]. Note that the problem of constructing a surface that is perpendicular to a measured gradient vector field is more common in the literature and is used in the measurement of corneal topography [7] and in determining shape from shading [8]. For the design of a driver-side mirror for an automobile note that flat mirrors provide a field of view in the range of 15° to 20°, depending on the size of the mirror and the position of the driver. The familiar solution to the blind-spot problem used by trucks and buses, where the problem is more acute, is to employ spherical mirrors, but these introduce considerable distortion. We seek a transformation \( T \) that will be a scaling map in the appropriate coordinates and therefore yield a perspective view. By choosing a large scaling constant we can increase the field of view as much as we like, but the distortion will also increase. We will see, though, that one can achieve as much as a 45° field of view without introducing much distortion.

We take the observer’s eye to be at the point \([a,0,0]\), and \([0,0,0]\) as the initial point of our mirror, and let the image plane be \( x = a - 1 \), as in Fig. 3. Assuming that the driver’s seat is on the left side of the car, then the ray from the driver’s eye to the origin will reflect at an angle of \( \psi \approx 65° \) to the driver’s left side, and travel a distance \( k \) until it strikes the object plane orthogonally. This implies that if the point \( p = [x,y,z] \) lies on a ray entering the driver’s eye, that this ray intersects the image plane at

\[
q = \left[ a - 1, \frac{y}{a-x}, \frac{z}{a-x} \right].
\]

(6)

For a flat mirror, the transformation \( T \) would transform \( q \) [in a sense made precise below in Eq. (7)] by a factor of \( k + a \) onto the object plane. The object plane is spanned by the vectors \( u = [-\sin(\psi), \cos(\psi), 0] \) and \( v = [0,0,1] \) and translated...

![Fig. 2. (Color online) Given a vector field \( N \), one may construct a surface by integrating radially out from an initial condition.](image)

![Fig. 3. Coordinates of the driver-side mirror problem.](image)
by the vector \( \mathbf{w} = [(k + a)\cos(\phi), -(k + a)\sin(\phi), 0] \). We wish to scale by a factor greater than \( k + a \), so we scale by a factor \( \lambda(k + a), \lambda > 1 \). Thus

\[
T(\mathbf{q}) = \frac{\lambda(k + a)y}{a - x} \mathbf{u} + \frac{\lambda(k + a)z}{a - x} \mathbf{v} + \mathbf{w}. \tag{7}
\]

This allows us to compute \( \mathbf{N}(x,y,z) \) as described above. Taking \( \lambda = 4 \) and \( k = 1000 \) (any large value will do in this case) will result in a \( T \) that corresponds to a mirror with an approximately 45° field of view for the driver.

Figure 4 depicts a prototype mirror based on the surface designed using our method. The upper image in Fig. 5 shows a view of a scene with a traditional flat driver-side mirror, while the lower image shows a view of the same scene using the prototype from Fig. 4. The mirror gives a 45° field of view, and so the blind spot has been eliminated, while the distortion is minimal.

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